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A Theory for the Mortality Curve of Filament Lamps

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We demonstrate for the first time that the ejection of tungsten atoms from a hot filament can be modeled by a binomial distribution. The relevant ejection chance per atom may have an arbitrary temporal profile depending on the presence of uniform Richardson evaporation, hot spots and temperature variation along the wire. A normal approximation is made and the lamp is supposed to fail if the undecayed atomic fraction drops below a critical value. The resulting formula for the lamp's survival probability in terms of error function has one free parameter and is shown to be in excellent agreement with the experimental mortality curve.

Keywords binomial distribution, half-life, mortality, normal approximation, survival probability, tungsten filament

1. Introduction

It is well-known for more than a century now that vacuum/gas-filled tungsten filament lamps nearing the end of their rated life often fail (Ref 1-5) in an apparently unpredictable manner. The possible physical mechanisms of the lamp's death may be attributed to tungsten evaporation and to manufacturing defects. The thermionic emission (Ref 4-12) has many components like spatially uniform Richardson–Dushman evaporation of tungsten atoms at some mean temperature, appearance of hot spots in the shape of hills/valleys, variation of temperature along the filament, etc. The manufacturing mechanisms (Ref 9, 13, 14) include non-uniform cross section of the wire, inhomogeneous dopant distribution, migration and growth of the potassium-filled bubbles within the wire, sliding of grain boundaries accompanied by torque generation, etc. The wire is supposed to snap at the thinnest spots due to preferential heating/mechanical weakness.

Furthermore, lighting engineers (Ref 3) have experimentally measured the so-called mortality curves of commercial bulbs showing their survival probability S as a function of the dimensionless time interval $\tau \equiv t/t_{1/2}$. Here t is the elapsed experimental time and $t_{1/2}$ the half-life for a fresh assembly of identical lamps kept in continuous operation. The measured points with few exceptions are known to lie on an essentially universal curve shown by the dashed line in Fig. 1. The curve varies rather slowly upto the half-life point $\tau = 1$ and drops rapidly thereafter. Clearly an exponential form $\exp(-\tau \ln 2)$ for S familiar in nuclear radioactivity cannot fit the data at all.

It is very surprising, however, that a mathematical theory for the mortality curve is still not available in the literature. The aim of the present article is to address this non-trivial issue based on the following crucial observations: (a) the universal shape of the dashed line in Fig. 1 is quite insensitive to the actual geometry/design parameters of specific bulbs; (b) the known mechanisms of lamp failure are essentially random statistical in nature; and (c) hence Fig. 1 should be deducible from purely stochastic arguments.

Section 2 below emphasizes the importance of the evaporation concept applicable to filaments and also points out that the undecayed number N of atoms should obey a binomial distribution at every time τ. Next, in Section 3 a normal approximation is made and the lamp is assumed to fail if the undecayed fraction of atoms drops below a critical value f_c.

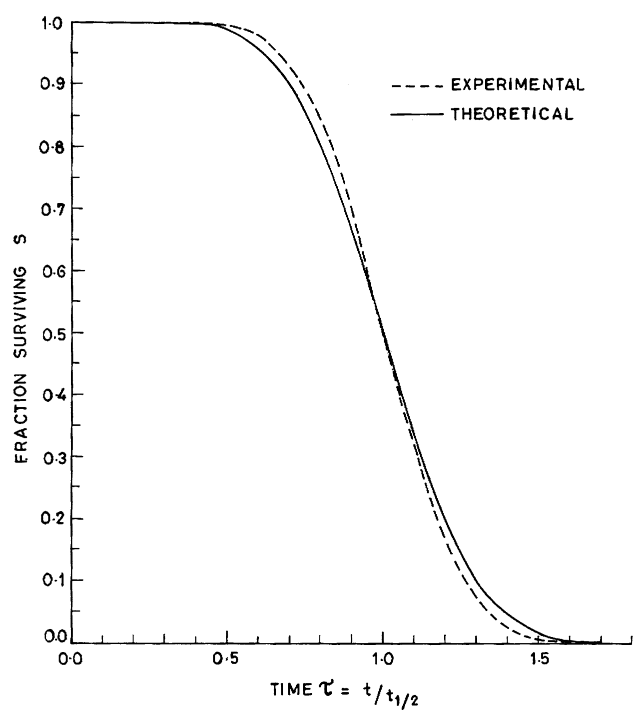

Fig. 1 Survival probability S of filament lamps. The dashed line corresponds to experimental observation (Ref 3) and the solid line pertains to our model (20)

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This allows us to derive an elegant expression for the survival probability S in terms of error functions containing one adjustable parameter Y related to the slope of the mortality curve at the half-life point. Next in Section 4, a numerical comparison between theory and experiment is done along with two interesting applications related to various measures of bulb's life and to the renewal rate of filament lamps. Finally, in Section 5 salient features of our work are summarized.

2. Hot Filament Evaporation and Stochastic Description

2.1 Importance of Tungsten Evaporation

It may be stressed that the concept of atomic thermionic effect in general and uniform Richardson evaporation in particular can satisfactorily explain several features of incandescent lamps operation. These features include mass loss of pure tungsten wire in vacuum (Ref 15), exponent – rules for life versus voltage (Ref 12, 16), relationship between temperature and life (Ref 11), hot-spot burn-out (Ref 9), etc. In all these treatments one starts from a filament in the state of full brilliance at zero time, having initial number of atoms $N(0)$ and temperature $T(0)$. Thereafter, at general time τ one sets up the deterministic evaporation and heat differential equations in presence of various components of the thermionic effects. In principle, one may like to solve these equations for obtaining the average observables of interest (labeled by a bar) namely the number of undecayed atoms $\overline{N}(\tau)$, the number of ejected atoms $\bar{n}(\tau) \equiv N(0) - \overline{N}(\tau)$, the evolving temperature $\overline{T}(\tau)$, etc. For our purpose it will be more convenient to employ the undecay chance per atom $\bar{f}(\tau)$ and ejection chance per atom $\bar{g}(\tau)$ defined over the interval τ by

$$\bar{f}(\tau) = \overline{N}/N(0); \quad \bar{g}(\tau) = \bar{n}/N(0); \quad \bar{f} + \bar{g} = 1 \qquad \text{(Eq 1)}$$

Of course, since the deterministic evaporation and heat equations are coupled non-linear it is impossible to solve them in analytically closed form valid at all times. However, some general properties of $\bar{f}$ and $\bar{g}$ are obvious namely $\bar{f}(0) = 1$, $\bar{f}(\infty) = 0$, and at intermediate times

$$1 > \bar{f} > 0; 0 < \bar{g} < 1 \text{ if } 0 < \tau < \infty \qquad \text{(Eq 2)}$$

Luckily, the explicit τ-dependent profiles of $\bar{f}$ and $\bar{g}$ will not be needed in the sequel.

2.2 Stochastic Description

Since thermionic effect is actually a random statistical process, the lamp observables also fluctuate with time around their average values and no overhead bars will be attached to the corresponding variates. Consider the event in which out of initial $N(0)$ atoms N atoms remain undecayed and $n \equiv N(0)-N$ atoms have been ejected till the time τ. Let us now give a simple argument for calculating the probability of such event. There are $\binom{N(0)}{N}$ different ways of selecting N undecayed atoms out of $N(0)$ initial ones. In every such selection the joint chance of having N survivors (i.e. successes) is $\bar{f}^N$, since atoms survive independently of each other. Also, in the same arrangement the joint chance of having n ejections (i.e. failures) is $\bar{g}^n$ since the atoms are emitted in mutually uncorrelated manner. Hence, the net probability of the aforesaid event becomes

$$Q_N = \binom{N(0)}{N} \bar{f}^N(\tau)\bar{g}^n(\tau) \qquad \text{(Eq 3)}$$

$$N = N(0), N(0) - 1, \ldots\ldots, 1, 0 \qquad \text{(Eq 4)}$$

which closely resembles the familiar binomial distribution (Ref 17). A more rigorous view of the event is to say that there exists a stochastic chain, in which the atoms 1, 2, ..., n are emitted at random times $\tau_1, \tau_2, \ldots\ldots \tau_n$. As proved in the Appendix a careful time-ordered multiple integration of the relevant conditional probability again leads to the formula (3). At any τ the statistical mean $\overline{N}(\tau)$ and standard deviation $\sigma(\tau)$ of the variate N read

$$\overline{N}(\tau) = N(0)\bar{f}; \quad \sigma(\tau) = \left\{N(0)\bar{f}\bar{g}\right\}^{1/2} \qquad \text{(Eq 5)}$$

2.3 Analysis of Standard Deviation (sd)

How do the different components of thermionic evaporation contribute to the net sd $\sigma(\tau)$? To answer this important question we write the ejected number as a physically meaningful additive variate $n = n_U + n_H + n_V$ where n_U accounts for spatially uniform Richardson emission at average temperature, n_H is contributed by the evaporation of a few hot spots, and n_V arises from small variations of temperature along the wire. In the sequel the symbol δ represents departure from the mean and expectation values are tacitly implied. Then the net variance and relative sd of n become

$$(\delta n)^2 = (\delta n_U)^2 + (\delta n_H)^2 + (\delta n_V)^2 + \text{ covariance terms}$$
$$\left|\frac{\delta n}{n}\right| \sim \left|\frac{\delta n_U}{n_U}\right| + \text{negligibles} \qquad \text{(Eq 6)}$$

because the wave symbol "~" represents order of magnitude, n_U dominates terribly over n_H or n_V, and similarly for $|\delta n_U|$. Equation (6) demonstrates that the relative sd of the ejected atoms is governed predominantly by the uniform Richardson component in the first (or leading) order since the original variate $n = n_U + n_H + n_V$ was additive.

It is worth investigating a naive, multiplicative form of the variate n_{naive} = constant. $n_U n_H n_V$. Its logarithmic differentiation leads to

$$\delta n_{naive}/n_{naive} = \delta n_U/n_U + \delta n_H/n_H + \delta n_V/n_V \qquad \text{(Eq 7a)}$$

Squaring and taking expectation values one arrives at the relative sd

$$\left|\frac{\delta n}{n}\right|_{naive} = \left\{\left(\frac{\delta n_U}{n_U}\right)^2 + \left(\frac{\delta n_H}{n_H}\right)^2 + \left(\frac{\delta n_V}{n_V}\right)^2 + \text{ covariance terms}\right\}^{\frac{1}{2}}$$

$$\approx \left\{\left(\frac{\delta n_H}{n_H}\right)^2 + \left(\frac{\delta n_V}{n_V}\right)^2 + 2\left(\frac{\delta n_H}{n_H}\right)\left(\frac{\delta n_V}{n_V}\right)\right\}^{\frac{1}{2}} \qquad \text{(Eq 7b)}$$

because $\delta n_H/n_H$ and $\delta n_V/n_V$ dominate terribly over $\delta n_U/n_U$ which is about 10^{-6} at 2700 K. The precise numerical value of the right-hand-side of (7b) depends on the sign of the last covariance term. However, standard correlation theory of mathematical statistics yields lower and upper bounds through the inequality

$$\left|\left|\frac{\delta n_H}{n_H}\right| - \left|\frac{\delta n_V}{n_V}\right|\right| < \left|\frac{\delta n}{n}\right|_{\text{naive}} < \left|\left|\frac{\delta n_H}{n_H}\right| + \left|\frac{\delta n_V}{n_V}\right|\right| \qquad \text{(Eq 7c)}$$

Unfortunately, (7c) cannot imply that the relative sd of the ejected number is governed by hot spots and/or temperature variation along the wire because the multiplicative model n_{naive} = const. $n_U n_H n_V$ is physically meaningless. Similarly for the manufacturing defects.

3. The Normal Approximation

3.1 Preliminaries

Since N takes up discrete integer values the binomial distribution (3) is not quite suitable for doing algebraic manipulations (e.g. calculating average values of physical observables). It is more convenient to use the continuous normal approximation (Ref 17) remembering that $N(0)$ is very large compared to unity, ala Avogadro. Then, at the end of τ, the differential probability of finding the number of undecayed atoms between N and $N-dN$ becomes

$$dQ = \frac{dN}{(2\pi)^{1/2}\sigma}\exp\left\{-\frac{\left(\overline{N}-N\right)^2}{2\sigma^2}\right\}; \quad N(0) \geq N \geq 0 \qquad \text{(Eq 8)}$$

$$= \frac{dw}{\pi^{1/2}}\exp\left(-w^2\right); \quad -\infty < w < \infty \qquad \text{(Eq 9)}$$

where we have introduced the transformation

$$w = \frac{\overline{N}-N}{2^{1/2}\sigma} = \left\{\frac{N(0)}{2\overline{f}\bar{g}}\right\}^{1/2}\left(\overline{f}-f\right) \qquad \text{(Eq 10)}$$

Some useful remarks are in order at this stage. (i) In (10) the basic random variable is the fraction f of undecayed atoms. (ii) In statistical literature the symbol $z \equiv \sqrt{2}w$ is called a standard normal variate. (iii) Strictly speaking, since $1 \geq f \geq 0$ the range of w ought to be written as

$$-\left\{N(0)\bar{g}/2\overline{f}\right\}^{1/2} \leq w \leq \left\{N(0)\overline{f}/2\bar{g}\right\}^{1/2} \qquad \text{(Eq 11)}$$

But at intermediate times this essentially coincides with the range $-\infty < w < \infty$ because $N(0)$ is large and $\overline{f}$ and $\bar{g}$ are proper fractions in the inequality (2).

3.2 The Critical Configuration

As evaporation proceeds and filament gets thinner, we expect the lamp to fail if the random fraction f drops below some critical value f_c, say. Physically such failure is attributed to preferential heating/mechanical weakness at the thinnest spot; experimentally the measured values (Ref 9) of the critical fraction f_c are around 85–90%. In our model the criterion for the bulb's survival is specified by the inequality

$$1 \geq f \geq f_c; \quad -\infty \leq w \leq w_c \qquad \text{(Eq 12)}$$

$$w_c(\tau) \equiv \left\{\frac{N(0)}{2\overline{f}\bar{g}}\right\}^{1/2}\left(\overline{f}-f_c\right) \qquad \text{(Eq 13)}$$

It may be emphasized that the critical fraction f_c is not an unknown free parameter of the theory. Indeed, we shall show in (19) below that $f_c = \overline{f}(\tau = 1)$.

3.3 General Survival Probability

During the interval τ the lamp's survival probability S is simply the cumulative integral of the distribution (9) over the truncated range defined by (12), i.e.,

$$\begin{aligned} S(\tau) &\equiv \int_{\text{trunc}} dQ = \frac{1}{\pi^{1/2}}\int_{-\infty}^{w_c} dw\exp\left(-w^2\right) \\ &= 0.5\{1+\text{erf}(w_c)\} \end{aligned} \qquad \text{(Eq 14)}$$

where erf denotes the Gauss error function (Ref 18). The general time-derivative of (14) labeled by a dot reads

$$\dot{S}(\tau) \equiv \frac{dS}{d\tau} = \pi^{-\frac{1}{2}}\dot{w}_c\exp\left(-w_c^2\right) \qquad \text{(Eq 15)}$$

As they stand (14, 15) still seem to depend on w_c of (13), i.e. on the temporal profiles of $\overline{f}$ and $\bar{g}$.

3.4 Restriction at the Half-Life Point

It is, however, possible to get rid of the said dependence on temporal profiles by judiciously imposing some restrictions at the half-life point $t = t_{1/2}$ or $\tau = 1$. For τ close to unity we can employ a linear expansion

$$w_c(\tau) \approx w_c(1) + (1-\tau)Y + \cdots \qquad \text{(Eq 16)}$$

where $-Y \equiv \dot{w}_c$ (1) is the first Taylor coefficient. Insertion of this w_c into (14, 15) and evaluation at the half life point itself yield.

$$S(1) = 0.5\{1+\text{erf}(w_c(1))\} = 0.5 \qquad \text{(Eq 17)}$$

$$\dot{S}(1) = \pi^{-\frac{1}{2}}\dot{w}_c(1)\exp\left\{-w_c^2(1)\right\} \qquad \text{(Eq 18)}$$

These conditions, in turn, lead to

$$w_c(1) = 0; \quad f_c = \overline{f}(1); \quad Y \equiv -\dot{w}_c(1) = -\pi^{1/2}\dot{S}(1) \qquad \text{(Eq 19)}$$

implying that the critical atomic fraction $\bar{f}_c$ is just the deterministic profile $\overline{f}(\tau)$ evaluated at the half life point. Furthermore, upon reading the numerical slope $\dot{S}(1)$ of the dashed curve in Fig. 1 and using (19) we get $Y = 3.0$. Hence, combining (14, 16) we arrive at the final expression for the lamp's survival probability at intermediate times namely

$$S(\tau) \cong 0.5\{1+\text{erf}(Y(1-\tau))\}; \quad Y = 3.0 \qquad \text{(Eq 20)}$$

3.5 Validity Domain of Eq (20)

One may argue that since a linear Taylor expansion of $w_c(\tau)$ in (16) should hold in the vicinity of the half life point (i.e. for $|u| \ll 1$) hence our approximate $S(\tau)$ in (20) may become unreliable far away from the half-life point (i.e. for $|u| \geq 1$) where $u \equiv 1 - \tau$. To answer this criticism we consider separately two distinct regions of the time variable:

(i) Let t be sufficiently *smaller* than $t_{1/2}$ such that $Y(1-\tau) \approx 3u > 1$, i.e. $u > 1/3$. Then the error function in (20) becomes close to unity, i.e. our approximate $S(\tau)$ becomes nearly 1 in agreement with the experimental curve.
(ii) Next, let t be sufficiently larger than $t_{1/2}$ such that $Y(1-\tau) \approx -3u < -1$. Then the error function in (20) becomes close to – 1, i.e. our approximate $S(\tau)$ reduces almost to zero, again in conformity with experiment. Consequently the expression (20) can be relied upon even if $|u| \geq 1$.

4. Numerical Work and Applications

4.1 Numerical Work on S(τ)

The solid curve in Fig. 1 is our theoretical prediction of the survival probability computed from (20). The solid and dashed lines agree accurately (both in value and in slope) at the half-life point $\tau = 1$, as they should by construction. In fact, the matching between the two curves is very good throughout the time domain of interest. This lends support to the physical/ mathematical arguments presented in Sections 1–3 above. Next, a detailed and a brief application of our $S(\tau)$ curve are described by items 4.2 and 4.3, respectively, below.

4.2 Various Types of Lives for Bulbs

The "life" shown on the packages of commercial incandescent lamps is actually the *half-life* $t_{1/2}$ in hours. More precisely, if an assembly of identical fresh lamps is switched on at the instant $t = 0$ then, on the average, only half their number would survive till the instant $t_{1/2}$. Experimentally, $t_{1/2}$ is measured by engineers through destructive testing. Theoretically it can be estimated by using the formula $t_{1/2} \approx M(0)/J(0)\,A(0)$ where, at the initial instant, the filament had mass $M(0)$, Richardson–Dushman evaporation rate $J(0)$, and surface area $A(0)$. Agrawal and Menon (Ref 11) have verified this formula in practice for a whole class of manufactured lamps in the wattage range 6–10,000 watts. This $t_{1/2}$ is crucial for constructing the dimensionless interval $\tau \equiv t/t_{1/2}$ used in the study of mortality statistics.

A nice application of our expression (20) for $S(\tau)$ is in the calculation of the instantaneous decay rate $R(\tau)$ and the related most probable life $t_m \equiv \tau_m t_{1/2}$. These extra symbols are defined mathematically by

$$R(\tau) \equiv -\dot{S}(\tau) = \left(Y/\sqrt{\pi}\right)\exp\left(-w_c^2\right) \qquad \text{(Eq 21)}$$

$$R(\tau_m) = \text{peak value;} \quad \dot{R}(\tau_m) = -\left(2Y^2/\sqrt{\pi}\right)w_m \exp\left(-w_m^2\right) = 0$$

where $w_c = Y(1-\tau)$ and $w_m = Y(1-\tau_m)$. For visual inspection we plot $R(\tau)$ in Fig. 2 exhibiting a peak at $w_m = 0$, i.e. $\tau_m = 1$. Physically this result implies that the half and most probable lives are the same in our model.

Also of interest is the calculation of the *average life* $t_{av} = \tau_{av} t_{1/2}$ defined mathematically by

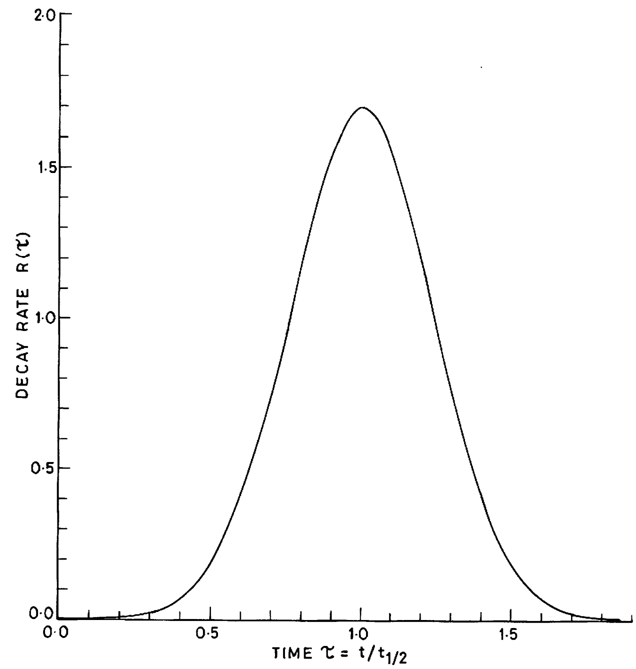

Fig. 2 The decay rate $R(\tau)$ of filament lamps as predicated by Eq. (21)

$$\begin{aligned}
\tau_{av} &\equiv \int_0^\infty d\tau\ \tau R(\tau) \Big/ \int_0^\infty d\tau R(\tau) \\
&= \int_{-\infty}^{Y} \frac{dw_c}{Y}\left(1 - \frac{w_c}{Y}\right)\cdot\frac{Y}{\sqrt{\pi}}e^{-w_c^2} \\
&= \frac{1}{2}\{1 + \text{erf}(Y)\} + \frac{e^{-Y^2}}{2Y\sqrt{\pi}} \\
&= 1 - O\left(e^{-Y^2}\right) = 1 - O\left(10^{-4}\right)
\end{aligned} \qquad \text{(Eq 22)}$$

where use has been made of the transformation $\tau = 1 - w_c/Y$ along with the definition (21) of $R(\tau)$. Hence, within a tiny correction term of order 10^{-4}, the half and average lives also coincide. Our result $t_{1/2} = t_m = t_{av}$ is essentially exact, obtained analytically using a logical expression of $S(\tau)$. In contrast, some preliminary investigations (Refs 20, 21) had also found $t_m = 1.03t_{1/2}$, $t_{av} = 0.99\ t_{1/2}$ within correction terms of order 10^{-2}, but their procedure was numerical based on an empirical formula $S(\tau) = \exp(-\ln 2.\tau^5)$.

4.3 Renewal Rate of Filament Lamps

Another nice application of our formula (20) can be done to the following problem. If an organization has a large installation of fresh lamps kept under continuous operation, then what is the mean renewal rate of fused bulbs after a periodic (or repeated) elapse of a chosen interval of time. To answer this interesting but non-trivial question one may have to employ the survival probability $S(\tau)$ as well as the decay probability $D(\tau) \equiv 1 - S(\tau)$ for setting up a recursion relation applicable to successive stages. This issue is under active investigation and the results will be communicated shortly.

5. Conclusions

Before ending we summarize the salient features of the present model:

(i) The approach is based on the proper narration of lamp failure mechanisms and on sound principles of mathematical statistics.
(ii) The number of atoms ejected over interval τ is shown to obey a binomial distribution whose relative standard deviation is found to be dominated by uniform Richardson–Dushman evaporation.
(iii) The final expression (20) of the survival probability is insensitive to the actual geometry/design parameters of individual bulbs, and it is easy to compute in practice.
(iv) The evaporation-based theory fits very well the main (or first order) features of the experimentally observed mortality curves. This implies that manufacturing defects can have only fluctuating (or second order) effect on Fig. 1.
(v) Our model leads to the analytical finding that, for a given type of incandescent bulb, the half-life, most probable life and average life are exactly equal.

Appendix: Stochastic Chain Derivation of Eq (3)

Basic Decay Law for *j*th Atom

For an evaporating filament let us return to the time-dependent non-decayed and ejected atomic fractions $\overline{f}(\tau)$, $\overline{g}(\tau)$ defined by (1) of the text. The basic differential probability of the *j*th atom getting emitted during the time span (τ_j, $\tau_j + d\tau_j$) reads

$$d\overline{g}(\tau_j) = \dot{\overline{g}}(\tau_j)d\tau_j = \overline{f}(\tau_j)\{\dot{\overline{g}}(\tau_j)d\tau_j/\overline{f}(\tau_j)\} \qquad \text{(Eq A1)}$$

where the dot represents derivative with respect to the argument. This law (A1) has two important properties. It obviously equals the elementary chance $\overline{f}(\tau_j)$ that the *j*th atom survives till the instant τ_j multiplied by the conditional chance of its decay during $d\tau_j$. Also, its integral over any finite interval τ is

$$\int_0^\tau \dot{\overline{g}}(\tau_j)d\tau_j = \left[\overline{g}(\tau_j)\right]_0^\tau = \overline{g}(\tau) \qquad \text{(Eq A2)}$$

Special Sequential Decay Chain

Next, suppose the atoms are distinguishable and consider a special stochastic event such that the atoms 1, 2, 3, ... n are emitted in succession during the spans (τ_1, $\tau_1 + d\tau_1$), (τ_2, $\tau_2 + d\tau_2$), ... (τ_n, $\tau_n + d\tau_n$), respectively but the remaining N atoms called $n+1$, $n+2$,.... $N(0) \equiv n+N$ remain undecayed upto a chosen instant τ. Remembering the basic decay/survival law (A1) for any atom the joint differential probability for the said event becomes

$$dP_n = C\{d\overline{g}(\tau_1)\ldots\ldots d\overline{g}(\tau_n)\}\{\overline{f}(\tau)\ldots\ldots\overline{f}(\tau)\} \qquad \text{(Eq A3)}$$

Here C is a normalization constant, various atoms are assumed to be uncorrelated, the first curly bracket has n factors, the second curly bracket has N factors, and the imposed time ordering requirement is

$$0 \le \tau_1 \le \tau_2 \le \ldots\cdots \le \tau_n \le \tau \qquad \text{(Eq A4)}$$

General Sequential Decay Chain

Actually the $N(0)$ atoms are indistinguishable and the instants τ_1, $\tau_2, \ldots\ldots\ldots \tau_n$ themselves are random subject to the requirement (A4). Integrating (A3) over these random instants we obtain the general cumulative probability of the said event as

$$P_n = C\left\{\int_0^\tau \dot{\overline{g}}(\tau_1)d\tau_1 \int_{\tau_1}^\tau \dot{\overline{g}}(\tau_2)d\tau_2 \ldots\ldots\ldots \int_{\tau_{n-1}}^\tau \dot{\overline{g}}(\tau_n)d\tau_n\right\}\{\overline{f}(\tau)\}^N \qquad \text{(Eq A5)}$$

But by a well-known theorem (Ref 19) of calculus a multiple time-ordered integral of this type is $1/n!$ times the corresponding freely running integral viz.

$$P_n = \frac{C}{n!}\left\{\int_0^\tau \dot{\overline{g}}(\tau_1)d\tau_1 \ldots\ldots\ldots \int_0^\tau \dot{\overline{g}}(\tau_n)d\tau_n\right\}\{\overline{f}(\tau)\}^N \qquad \text{(Eq A6)}$$

$$= \frac{C}{n!}\{\overline{g}(\tau)\}^n\{\overline{f}(\tau)\}^N; \quad 0 \le n \le N(0) \qquad \text{(Eq A7)}$$

in view of the basic property (A2).

Normalization

Finally, imposing the condition of the total probability

$$\sum_{n=0}^{N(0)} P_n = 1 \qquad \text{(Eq A8)}$$

we identify $C = N(0)!/N!$ so that P_n of (A7) indeed coincides with Q_N of Eq. (3).

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